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L^1-Contraction and Uniqueness for Quasilinear Elliptic–Parabolic Equations

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We prove the L^1-contraction principle and uniqueness of solutions for quasilinear elliptic–parabolic equations of the form

[illegible]

weak solution of finite energy. In particular, we do *not* suppose that the distributional derivative $\partial_t[b(u)]$ is a bounded Borel measure or a locally integrable function.

1. INTRODUCTION

Let $b\colon R \to R$ be monotone nondecreasing and continuous and let $a\colon R^n \times R \to R^n$ s.t.

$$u \in H^{1,r}(\Omega) \mapsto -\operatorname{div}[a(\nabla u, w)] \in (H^{1,r}(\Omega))^*$$

is strictly monotone for any $w \in R$ and some $r \in (1, \infty)$. Then

$$\partial_t[b(u)] - \operatorname{div}[a(\nabla u, b(u))] + f(b(u)) = 0 \qquad \text{in} \quad (0, T) \times \Omega \tag{1}$$

is a quasilinear elliptic–parabolic equation in u, which also can be interpreted as a doubly nonlinear degenerate parabolic equation in $b(u)$. The problem is completed by prescribing some conditions for u on the boundary $\partial\Omega$ of Ω (Dirichlet conditions on a subset S of $\partial\Omega$ and Neumann conditions elsewhere) and some initial values for $b(u)$:

$$u = u^D \qquad \text{on} \quad (0, T) \times S, \tag{2}$$

$$a(\nabla u, b(u)) \cdot \nu = 0 \qquad \text{on} \quad (0, T) \times (\partial\Omega - S), \tag{3}$$

$$b(u) = b^0 \qquad \text{on} \quad \{0\} \times \Omega. \tag{4}$$

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Alt and Luckhaus [1, Theorem 1.7] have shown that the natural solution space for this equation is given by all u of "finite energy," i.e.,

$$\sup_{t \in (0,T)} \int_{\Omega} \Psi(b(u(t))) + \int_{(0,T)\times\Omega} |\nabla u|^r < +\infty,$$

where Ψ is the Legendre transform of the primitive of b (see (8) for the definition).

The usual approach to the (generalized, as depending on the Lipschitz constant L of f) L^1-contraction principle, i.e.,

$$\int_{\Omega} |b(u_1(t)) - b(u_2(t))| \leqslant \exp(Lt) \int_{\Omega} |b_1^0 - b_2^0| \qquad \text{for} \quad t \in (0, T) \tag{5}$$

for any two solutions u_1 and u_2, is to multiply the difference of the equation for u_1 and that for u_2 with $\eta'_\delta(u_1 - u_2)$, where η'_δ is some smooth and monotone approximation of

$$\operatorname{sign}(z) = \left\{ \begin{array}{rl} 1 & \text{for} \quad z > 0 \\ 0 & \text{for} \quad z = 0 \\ -1 & \text{for} \quad z < 0 \end{array} \right\}.$$

This argument can be made rigorous for finite energy solutions u_1 and u_2 provided that

$$\partial_t[b(u_i)] \in L^1((0, T) \times \Omega), \tag{6}$$

see for instance [1, Theorem 2.2] or [3, Théorème 2.2; 22, Theorem 1] for some special cases. If b^{-1} is continuously differentiable, the above argument can be made rigorous for finite energy solutions u_1 and u_2 with the additional property

$$b(u_i) \in BV((0, T) \times \Omega),$$

see Yin [24, Theorem 2.4].

As has been recently announced [15–17], we are able to prove (5) for solutions of finite energy *without* assuming (6). Our technique is inspired by a method introduced by Kružkov [13] to prove L^1-contraction for entropy solutions for scalar conservation laws; the doubling of variables. Carrillo [5] probably was the first to apply Kružkov's method to second order equations (see also [9]). But for our problem, it is more convenient to apply this procedure only to the time variable—the space variables are handled as in [1, Theorem 2.2]. In two forthcoming papers, [14] and [18], we apply our technique to some models beyond the above frame:

• a model for transport of reactive solutes in porous media with equilibrium and non-equilibrium multiple-site adsorption (see [7]). This is a system with time- and space-dependent coefficients of low regularity.

• a model for saturated–unsaturated water flow through porous media (see [2])—here, time-dependent unilateral boundary conditions enforce a formulation as a variational inequality.

Other techniques of proving uniqueness without passing by (5) have been developed, but they all seem restricted to the case of

$$a(p, w) = Ap + e(w) \qquad \text{with linear } A. \tag{7}$$

One idea is to interpret the difference of the equation for u_1 and that for u_2 as a linear parabolic equation in $b(u_1) - b(u_2)$ and then to solve the dual equation, see for instance Kamin [11, Theorem 1] in the case of smooth b^{-1} and one space dimension. This technique has been refined by de Pablo and Vazquez [19, Theorem 2.1] and by Peletier and Junning [20, Theorem 2.1] to cover the case of $b(z) = \operatorname{sign}(z)\,|z|^{1/m}$.

Brézis and Crandall [4] prove uniqueness in the case of $e(w) = 0$. Their idea is to apply $(\mathrm{id} - \epsilon\Delta)^{-1}$ to the difference of the equations. This technique even works if b is a maximal monotone graph.

A third approach consists in multiplying the difference of the equations with $\int_0^t (u_1 - u_2)(\tau)\, d\tau$, see Gilding [8, Theorem 1], Gilding and Peletier [10 or 1, Theorem 2.4]. Also, this technique is applicable if b is a maximal monotone graph, but in this case e in (7) must be s.t. $e(w_1) = e(w_2)$ for $w_1, w_2 \in b(z)$. If e depends nonlinearly on the jumps of b, one definitely needs some kind of entropy condition to guarantee uniqueness (see Vol'pert and Hudjaev [23]).

The semigroup approach is a way of circumventing the problem of uniqueness of weak solutions for the time–dependent equations: Simondon [21] constructs a T-accretive operator in $L^1(\Omega)$ s.t. the corresponding well-defined semigroup consists of weak solutions.

The reader will find many other references in [12].

2. ASSUMPTIONS AND NOTATIONS

Let us now give the full set of assumptions and the notations used; we essentially adopt the frame of [1, Theorem 2.2].

(1) $\Omega \subset R^n$ is open, connected and bounded with Lipschitz boundary; $S \subset \partial\Omega$ is $\mathscr{H}^{n-1}$-measurable with positive measure. u^D is assumed to

be in $H^{1,r}(\Omega) \cap L^\infty(\Omega)$. For convenience, we write $Q := (0, T) \times \Omega$ and $\Gamma^D := (0, T) \times S$. The closed subspace V of $H^{1,r}(\Omega)$ is given by

$$V := \{v \in H^{1,r}(\Omega) \mid v = 0 \ \mathscr{H}^{n-1}\text{-a.e. on } S\}.$$

(2) $b: R \to R$ is monotone nondecreasing and continuous. The Legendre transform Ψ of the primitive of b is given by

$$\Psi(w) := \sup_{z \in R} \left(zw - \int_0^z b(\zeta)\, d\zeta \right) \tag{8}$$

and hence is a convex and lower semicontinuous function $\Psi: R \to [0, +\infty]$.

We also list some properties of Ψ which are related to those we will prove for a more general object (see (23)): Up to a constant, Ψ is characterized by

$$\partial \Psi(w) = \{z \mid b(z) = w\}, \tag{9}$$

where ∂ denotes the subgradient, admits the representation

$$\Psi(b(z)) = zb(z) - \int_0^z b(\zeta)\, d\zeta \tag{10}$$

and is superlinear in the sense that for any $\delta > 0$, there exists a $C_\delta < +\infty$ s.t. for all $w \in R$

$$|w| \leqslant \delta \Psi(w) + C_\delta. \tag{11}$$

(3) $a: R^n \times R \to R^n$ and $f: R \to R$ are continuous and satisfy the following conditions, where r^* denotes the dual exponent of $r \in (1, \infty)$:

- Natural growth: there exists a $C < +\infty$ s.t. for all $p, w \in R$

$$|a(p, w)|^{r^*} + |f(w)|^{r^*} \leqslant C[1 + |p|^r + \Psi(w)].$$

- Strict monotonicity of $a(p, w)$ in p: there exists a $c > 0$ s.t. for all $p_1, p_2, w \in R$

$$(a(p_1, w) - a(p_2, w)) \cdot (p_1 - p_2) \geqslant c\, |p_1 - p_2|^r. \tag{12}$$

- Hölder continuity of $a(p, b(z))$ in z: there exists a $C < +\infty$ s.t. for all $p, z_1, z_2 \in R$

$$|a(p, b(z_1)) - a(p, b(z_2))|^{r^*} \leqslant C\, |z_1 - z_2|\, [1 + |p|^r + \Psi(b(z_1)) + \Psi(b(z_2))]. \tag{13}$$

We now give the exact definition of finite energy solution of (1), (2), (3) and (4).

DEFINITION. (a) u is called subsolution with initial data b^0 if

$$
\begin{aligned}
b^0 \in L^1(\Omega) \quad & \text{satisfies} \quad \Psi(b^0) \in L^1(\Omega), \\
u \in L^r((0,T), H^{1,r}(\Omega)) \quad & \text{satisfies} \quad \Psi(b(u)) \in L^\infty((0,T), L^1(\Omega)) \\
& \text{and} \quad u \leqslant u^D \ \mathscr{H}^n\text{-a.e. on } \Gamma^D
\end{aligned}
$$

and the weak differential inequality holds

$$
\int_Q \{(b^0 - b(u))\,\partial_t\zeta + a(\nabla u, b(u)) \cdot \nabla\zeta + f(b(u))\zeta\} \leqslant 0
$$

$$
\text{for all nonnegative } \zeta \in L^r((0,T), V) \text{ with } \partial_t\zeta \in L^\infty(Q) \text{ and } \zeta(T) = 0. \tag{14}
$$

(b) u is called supersolution with initial data b^0 if

$$
\begin{aligned}
b^0 \in L^1(\Omega) \quad & \text{satisfies} \quad \Psi(b^0) \in L^1(\Omega), \\
u \in L^r((0,T), H^{1,r}(\Omega)) \quad & \text{satisfies} \quad \Psi(b(u)) \in L^\infty((0,T), L^1(\Omega)) \\
& \text{and} \quad u \geqslant u^D \ \mathscr{H}^n\text{-a.e. on } \Gamma^D
\end{aligned}
$$

and the weak differential inequality holds

$$
\int_Q \{(b^0 - b(u))\,\partial_t\zeta + a(\nabla u, b(u)) \cdot \nabla\zeta + f(b(u))\zeta\} \geqslant 0
$$

for all nonnegative $\zeta \in L^r((0,T), V)$ with $\partial_t\zeta \in L^\infty(Q)$ and $\zeta(T) = 0$.

(c) We call u solution with initial data b^0 if u is both sub- and supersolution with respect to b^0.

For convenience, we write

$$
w^+ := \begin{cases} w & \text{for } w \geqslant 0 \\ 0 & \text{for } w \leqslant 0 \end{cases} \quad \text{and} \quad \operatorname{sign}^+ := \begin{cases} 1 & \text{for } w > 0 \\ 0 & \text{for } w \leqslant 0 \end{cases}.
$$

3. RESULTS

THEOREM. (a) *Let u_1 and u_2 be sub- resp. supersolution with initial data b_1^0 resp. b_2^0. Then*

$$\int_Q \{[(b_1^0 - b_2^0)^+ - (b(u_1) - b(u_2))^+]\, \partial_t \gamma$$
$$+ \operatorname{sign}^+(u_1 - u_2)[a(\nabla u_1, b(u_1)) - a(\nabla u_2, b(u_2))] \cdot \nabla \gamma$$
$$+ \operatorname{sign}^+(b(u_1) - b(u_2))[f(b(u_1)) - f(b(u_2))]\gamma\} \leqslant 0$$
$$\text{for all nonnegative} \quad \gamma \in C_0^\infty((-\infty, T) \times R^n). \tag{15}$$

In particular, if there exists an $L \in R$ s.t.

$$-(f(w_1) - f(w_2)) \leqslant L(w_1 - w_2) \qquad \text{for} \quad w_1 > w_2, \tag{16}$$

we obtain for a.e. $t \in (0, T)$

$$\int_\Omega (b(u_1(t)) - b(u_2(t)))^+ \leqslant \exp(Lt) \int_\Omega (b_1^0 - b_2^0)^+. \tag{17}$$

(b) *If* (16) *holds, then there exists a unique solution u to given initial data b^0.*

Observe that the nondecreasing and continuous function b defines a nonnegative Borel measure $b(d\zeta)$ which has no atoms. So the r.h.s. expression in (18) below is well defined.

LEMMA 1. *Let η: $R \to R$ have bounded and continuous first and second derivatives. Let q: $R \times R \to R$ be related to η by*

$$q(z, z^0) = \int_{z^0}^{z} \eta'(\zeta - z^0)\, b(d\zeta). \tag{18}$$

(a) *Let u be a subsolution and suppose η is nondecreasing. Let $v^0 \in H^{1,r}(\Omega)$ be s.t. $\Psi(b(v^0)) \in L^1(\Omega)$ and*

$$\eta'(u - v^0) = 0 \quad \mathscr{H}^n\text{-a.e. on } \Gamma^D. \tag{19}$$

Then

$$\int_Q \{-q(u, v^0)\, \partial_t \gamma + a(\nabla u, b(u)) \cdot \nabla[\eta'(u - v^0)\gamma] + f(b(u))\, \eta'(u - v^0)\gamma\} \leqslant 0$$
$$\text{for all nonnegative} \quad \gamma \in C_0^\infty((0, T) \times R^n). \tag{20}$$

(b) *Let u be a supersolution and suppose η is nonincreasing. Let $v^0 \in H^{1,r}(\Omega)$ be s.t. $\Psi(b(v^0)) \in L^1(\Omega)$ and $\eta'(u-v^0)=0$ $\mathscr{H}^{:n}$-a.e. on Γ^D. Then*

$$\int_Q \{-q(u,v^0)\,\partial_t\gamma + a(\nabla u, b(u))\cdot\nabla[\eta'(u-v^0)\gamma] + f(b(u))\,\eta'(u-v^0)\gamma\} \leqslant 0$$

for all nonnegative $\gamma \in C_0^\infty((0,T)\times R^n)$. (21)

LEMMA 2. (a) *Let u be a subsolution with initial data b^0. Then*

$$\operatorname*{ess\,lim}_{t\downarrow 0} \int_\Omega (b(u(t)) - b^0)^+ = 0. \tag{22}$$

(b) *Let u be a supersolution with initial data b^0. Then*

$$\operatorname*{ess\,lim}_{t\downarrow 0} \int_\Omega (b^0 - b(u(t)))^+ = 0.$$

4. PROOF

Let us mention the main ingredients of the proof. The first step is Lemma 1, which is a kind of "entropy condition": Observe that by (18), the map $z \mapsto q(z, z^0)$ is indeed the entropy flux for the (possibly nonconvex) entropy $z \mapsto \eta(z-z^0)$ with respect to b. Notice that conventional entropy conditions—from a mathematical point of view—express the validity of the chain rule (at least as an inequality) for weak solutions with respect to the dependent variable(s). Lemma 1 just tells us that—although (6) may not be true—the chain rule

$$\langle \partial_t[b(u)], \eta'(u-v^0)\gamma\rangle = -\int_Q q(u,v^0)\,\partial_t\gamma$$

still holds for a solution u of finite energy, where the pairing is in $L^2((0,T),V)^* \times L^2((0,T),V)$.

Once we have this "entropy condition", we derive (15) with help of a method introduced by Kružkov [13] to prove L^1-contraction for entropy solutions of scalar conservation laws: "variable doubling". As for "viscosity solutions" of first order equations (in the sense of [6]), this procedure overcomes the lack of regularity of the solutions: it allows to treat u_2 as a constant with respect to the differential equation of u_1—and vice versa. But due to the second order term in space, we cannot carry out the full program of variable doubling, we only double the t-variable: it permits us at least to treat u_2 as independent of the time variable of u_1—and vice versa. This restriction of the variable doubling procedure to the time variable is in some sense compensated by the regularity of the solutions in the other variables: $\nabla u \in L^2(Q)$. Observe that Lemma 1 accounts for this partial application: v^0 is not restricted to the class of constants as in conventional entropy conditions but is allowed to depend on the space variables.

Let us start with the proof of Lemma 1. As the proof of part b) is similar, we restrict ourselves to a). Because of the linearity in η, we may assume that η is convex. We mimic the definition of Ψ in (8) to introduce a transform η^* of η. For $z^0 \in R$, we define the convex and lower semicontinuous $\eta^*(\cdot, z^0)$: $R \mapsto (-\infty, +\infty]$ by

$$\eta^*(w, z^0) := \sup_{z \in R} \left\{ \eta'(z - z^0)(w - b(z)) + \int_{z^0}^{z} \eta'(\zeta - z^0)\, b(d\zeta) \right\}. \tag{23}$$

Analogous to (9), we have

$$\partial \eta^*(w, z^0) \supset \{ \eta'(z - z^0) \mid b(z) = w \}, \tag{24}$$

which by definition is equivalent to

$$\eta^*(w, z^0) - \eta^*(b(z), z^0) \geqslant \eta'(z - z^0)(w - b(z))$$

for arbitrary $w, z \in R$. This follows from

$$\begin{aligned} &\eta^*(w, z^0) - \eta^*(b(z), z^0) \\ &\quad \geqslant \inf_{\tilde z \in R} \left\{ \eta'(z - z^0)(w - b(z)) - \eta'(\tilde z - z^0)(b(z) - b(\tilde z)) + \int_{\tilde z}^{z} \eta'(\zeta - z^0)\, b(d\zeta) \right\} \end{aligned}$$

and

$$\int_{\tilde z}^{z} \eta'(\zeta - z^0)\, b(d\zeta) \geqslant \eta'(\tilde z - z^0)(b(z) - b(\tilde z)), \tag{25}$$

the last inequality being a consequence of the monotonicity of b and η'. (24) visualizes that $\eta^*(\cdot, z^0)$ needs not to be differentiable if b is not strictly increasing. (23) and (25) also yield the following representation of η^* in the spirit of (10)

$$\eta^*(b(z), z^0) = \int_{z^0}^{z} \eta'(\zeta - z^0)\, b(d\zeta) = q(z, z^0). \tag{26}$$

In view of this, Lemma 1 is proved if we can show that (24) implies

$$\int_Q \{[\eta^*(b^0, v^0) - \eta^*(b(u), v^0)]\, \partial_t \gamma$$
$$+ a(\nabla u, b(u)) \cdot \nabla[\eta'(u - v^0)\gamma] + f(b(u))\, \eta'(u - v^0)\gamma\} \leqslant 0$$
$$\text{for all nonnegative} \quad \gamma \in C_0^\infty((-\infty, T) \times R^n), \tag{27}$$

which is essentially one half of the chain rule

$$\langle \partial_t [b(u)], \eta'(u - v^0)\gamma \rangle = -\int_Q \eta^*(b(u), v^0)\, \partial_t \gamma.$$

This will be now done with help of a technique introduced by Alt and Luckhaus [1, Lemma 1.5]. The boundedness of η'' and (19) ensure that

$$\zeta := \eta'(u - v^0)\gamma \in L^r((0, T), V).$$

As a consequence of the boundedness and nonnegativity of η',

$$\zeta_h(t) := \frac{1}{h} \int_t^{t+h} \zeta(\tau)\, d\tau$$

defines an admissible testfunction in (14) for any $h > 0$. Let us consider the limit $h \downarrow 0$. For notational convenience, we extend b(u) to the negative time axis by

$$b(u(t)) = b^0 \qquad \text{for} \quad t < 0.$$

Using partial summation

$$\int_Q (b^0 - b(u))\, \partial_t \zeta_h = \int_{(0, T)} \int_\Omega (b^0 - b(u(t))) \frac{1}{h} \{\zeta(t + h) - \zeta(t)\}\, dt$$
$$= \int_{(0, T)} \int_\Omega \frac{1}{h} \{b(u(t)) - b(u(t - h))\}\, \zeta(t)\, dt,$$

the relation (24)

$$\{b(u(t)) - b(u(t - h))\}\, \zeta(t) = \{b(u(t)) - b(u(t - h))\}\, \eta'(u(t) - v^0)\, \gamma(t)$$
$$\geqslant \{\eta^*(b(u(t)), v^0) - \eta^*(b(u(t - h)), v^0)\}\, \gamma(t),$$

and again partial summation

$$\int_{(0,T)} \int_\Omega \frac{1}{h} \{\eta^*(b(u(t)), v^0) - \eta^*(b(u(t-h)), v^0)\} \, \gamma(t) \, dt$$

$$= \int_{(0,T)} \int_\Omega [\eta^*(b^0, v^0) - \eta^*(b(u(t)), v^0)] \frac{1}{h} \{\gamma(t+h) - \gamma(t)\} \, dt,$$

we obtain

$$\inf_{h \downarrow 0} \lim \int_Q (b^0 - b(u)) \, \partial_t \zeta_h \geqslant \int_Q [\eta^*(b^0, v^0) - \eta^*(b(u), v^0)] \, \partial_t \gamma.$$

Because of

$$\zeta_h \to \zeta \qquad \text{in} \quad L^r((0,T), H^{1,r}(\Omega)),$$

we infer for the remaining terms of the r.h.s. of (14)

$$\lim_{h \downarrow 0} \int_Q \{a(\nabla u, b(u)) \cdot \nabla \zeta_h + f(b(u)) \zeta_h\}$$

$$= \int_Q \{a(\nabla u, b(u)) \cdot \nabla[\eta'(u - v^0) \gamma] + f(b(u)) \, \eta'(u - v^0) \gamma\}.$$

This establishes (27) and therefore Lemma 1.

Let us now prove Lemma 2. Because of the obvious symmetry, we only consider part a). Fix a convex and smooth η s.t.

$$\eta(z) = 0 \qquad \text{for} \quad z \leqslant 0 \qquad \text{and} \qquad \eta(z) = z - \tfrac{1}{2} \qquad \text{for} \quad z \geqslant 1. \tag{28}$$

Consider

$$\eta_\delta(z) := \delta \eta \left(\frac{z}{\delta}\right)$$

and the related η_δ^* given by (23). The representation (26) yields the estimate

$$0 \leqslant (b(z) - b(z^0))^+ - \eta_\delta^*(b(z), z^0) \leqslant b(z^0 + \delta) - b(z^0).$$

By lower semicontinuity of $\eta^*(\cdot, z^0)$ and uniform continuity of b on bounded intervals,

$$\eta^*_\delta(w, z^0) \leqslant (w - b(z^0))^+ \qquad \text{for} \quad w \in \overline{b(R)} \quad \text{and} \quad z^0 \in R, \tag{29}$$

$$\eta^*_\delta(b(z), z^0) \to (b(z) - b(z^0))^+ \qquad \left\{\begin{array}{l}\text{for } \delta \downarrow 0 \text{ uniformly in} \\ z \in R \text{ and bounded } z^0.\end{array}\right\} \tag{30}$$

Let v^0 be s.t.

$$\begin{gathered} v^0 \in H^{1,r}(\Omega) \cap L^\infty(\Omega), \\ \Psi(b(v^0)) \in L^1(\Omega) \qquad \text{and} \qquad v^0 = u^D \quad \mathscr{H}^{n-1}\text{-a.e. on } S. \end{gathered} \tag{31}$$

Obviously, η_δ and v^0 are admissible in (27) for fixed $\delta > 0$. In particular, we obtain the one-dimensional weak differential inequality

$$\begin{aligned} &-\int_{(0,T)} \alpha'(t) \int_\Omega [\eta^*_\delta(b(u(t)), v^0) - \eta^*_\delta(b^0, v^0)]\, dt \\ &\qquad \leqslant \int_{(0,T)} \alpha(t)\, \theta(t)\, dt \qquad \text{for all nonnegative} \quad \alpha \in C_0^\infty((-\infty, T)) \end{aligned}$$

for some $\theta \in L^1((0, T))$. Thus there exists a set $E \subset (0, T)$ of zero measure s.t.

$$\limsup_{t \downarrow 0,\, t \notin E} \int_\Omega \eta^*_\delta(b(u(t)), v^0) \leqslant \int_\Omega \eta^*_\delta(b^0, v^0). \tag{32}$$

On the other hand, as $b^0 \in \overline{b(R)}$ a.e. in Ω (due to $\Psi(b^0) \in L^1(\Omega)$ and $\Psi = +\infty$ on $R - \overline{b(R)}$) and $v^0 \in L^\infty(\Omega)$, we infer from (29) resp. (30)

$$\int_\Omega \eta^*_\delta(b^0, v^0) \leqslant \int_\Omega (b^0 - b(v^0))^+, \tag{33}$$

$$\int_\Omega \eta^*_\delta(b(u(t)), v^0) \to \int_\Omega (b(u(t)) - b(v^0))^+ \quad \left\{\begin{array}{l}\text{for } \delta \downarrow 0 \text{ uniformly} \\ \text{in } t \in (0, T).\end{array}\right\} \tag{34}$$

Observe that we may choose the set $E \subset (0, T)$ in (32) to be independent of k for some fixed sequence $\delta_k \downarrow 0$. So we deduce from (32), (33) and (34)

$$\limsup_{t \downarrow 0,\, t \notin E} \int_\Omega (b(u(t)) - b(v^0))^+ \leqslant \int_\Omega (b^0 - b(v^0))^+.$$

The reader might now convince himself that b^0 can be approximated in $L^1(\Omega)$ by $b(v^0)$, where the v^0 have the properties in (31). Once again, E can be considered the same for all the v^0 of this approximating sequence. This proves (22).

Let us now derive the weak differential inequality (15) by doubling the time variable:

$$(t_1, t_2, x) \in (0, T)^2 \times \Omega := \tilde{Q}.$$

With a slight abuse of notation, we extend u_1 and u_2 to $\tilde{Q}$ by

$$u_1(t_1, t_2, x) := u_1(t_1, x) \qquad \text{and} \qquad u_2(t_1, t_2, x) := u_2(t_2, x). \tag{35}$$

The first step towards (15) will be the derivation of

$$\begin{aligned}\int_{\tilde{Q}} \{ &-(b(u_1) - b(u_2))^+ \, (\partial_{t_1} + \partial_{t_2})\gamma \\ &+ \operatorname{sign}^+(u_1 - u_2)[a(\nabla u_1, b(u_1)) - a(\nabla u_2, b(u_2))] \cdot \nabla\gamma \\ &+ \operatorname{sign}^+(u_1 - u_2)[f(b(u_1)) - f(b(u_2))]\gamma\} \leqslant 0 \\ &\text{for all nonnegative} \quad \gamma \in C_0^\infty((0, T)^2 \times R^n).\end{aligned} \tag{36}$$

Fix a smooth and convex η with the properties (28) and consider

$$\eta_\delta^+(z) := \delta\eta\left(\frac{z}{\delta}\right) \qquad \text{and} \qquad \eta_\delta^-(z) := \delta\eta\left(-\frac{z}{\delta}\right).$$

Obviously, η_δ^+, $v^0 = u_2(t_2)$ and $\{\gamma(t_2) \colon (t, x) \mapsto \gamma(t, t_2, x)\}$ are admissible in (20) for fixed $\delta > 0$ and a.e. $t_2 \in (0, T)$, yielding

$$\begin{aligned}\int_Q \{ &-q_\delta^+(u_1, u_2(t_2)) \, \partial_t \gamma(t_2) \\ &+ a(\nabla u_1, b(u_1)) \cdot \nabla[(\eta_\delta^+)' \, (u_1 - u_2(t_2)) \, \gamma(t_2)] \\ &+ f(b(u_1))(\eta_\delta^+)' \, (u_1 - u_2(t_2)) \, \gamma(t_2)\} \leqslant 0,\end{aligned} \tag{37}$$

where q_δ^+ is related to η_δ^+ by (18). Likewise, we obtain from (21)

$$\begin{aligned}\int_Q \{ &-q_\delta^-(u_2, u_1(t_1)) \, \partial_t \gamma(t_1) \\ &+ a(\nabla u_2, b(u_2)) \cdot \nabla[(\eta_\delta^-)' \, (u_2 - u_1(t_1)) \, \gamma(t_1)] \\ &+ f(b(u_2))(\eta_\delta^-)' \, (u_2 - u_1(t_1)) \, \gamma(t_1)\} \leqslant 0\end{aligned} \tag{38}$$

for a.e. $t_1 \in (0, T)$. We now integrate (37) over $t_2 \in (0, T)$ and (38) over $t_1 \in (0, T)$, add both resulting inequalities, use $(\eta_\delta^-)'(z) = -(\eta_\delta^+)'(-z)$ and end up with

$$\begin{aligned}\int_{\tilde{Q}} \{ &-[q_\delta^+(u_1, u_2)\, \partial_{t_1}\gamma + q_\delta^-(u_2, u_1)\, \partial_{t_2}\gamma] \\ &+ [a(\nabla u_1, b(u_1)) - a(\nabla u_2, b(u_2))] \cdot \nabla[(\eta_\delta^+)'(u_1 - u_2)\gamma] \\ &+ [f(b(u_1)) - f(b(u_2))](\eta_\delta^+)'(u_1 - u_2)\gamma\} \leqslant 0, \end{aligned} \tag{39}$$

where we now use the more convenient notation (35). The term coming from the second order operator involves second derivatives of η_δ and thus is singular in the limt $\delta \downarrow 0$. But due to the convexity of η_δ, the singular part has positive sign, as we will see:

$$\begin{aligned}&[a(\nabla u_1, b(u_1)) - a(\nabla u_2, b(u_2))] \cdot \nabla[(\eta_\delta^+)'(u_1 - u_2)\gamma] \\ &\quad = (\eta_\delta^+)'(u_1 - u_2)[a(\nabla u_1, b(u_1)) - a(\nabla u_2, b(u_2))] \cdot \nabla\gamma \\ &\qquad + [a(\nabla u_1, b(u_1)) - a(\nabla u_2, b(u_2))] \cdot (\nabla u_1 - \nabla u_2)(\eta_\delta^+)''(u_1 - u_2)\gamma\end{aligned}$$

and

$$\begin{aligned}&[a(\nabla u_1, b(u_1)) - a(\nabla u_2, b(u_2))] \cdot (\nabla u_1 - \nabla u_2)(\eta_\delta^+)''(u_1 - u_2) \\ &\quad = [a(\nabla u_1, b(u_2)) - a(\nabla u_2, b(u_2))] \cdot (\nabla u_1 - \nabla u_2)(\eta_\delta^+)''(u_1 - u_2) \\ &\qquad + [a(\nabla u_1, b(u_1)) - a(\nabla u_1, b(u_2))] \cdot (\nabla u_1 - \nabla u_2)(\eta_\delta^+)''(u_1 - u_2) \\ &\quad \geqslant c\, |\nabla u_1 - \nabla u_2|^r\, (\eta_\delta^+)''(u_1 - u_2) \\ &\qquad - C\, |u_1 - u_2|^{1/r^*}\, [1 + |\nabla u_1|^r + \Psi(b(u_1)) + \Psi(b(u_2))]^{1/r^*} \\ &\qquad \times |\nabla u_1 - \nabla u_2|\, (\eta_\delta^+)''(u_1 - u_2) \\ &\quad \geqslant -C\, |u_1 - u_2|\, [1 + |\nabla u_1|^r + \Psi(b(u_1)) + \Psi(b(u_2))](\eta_\delta^+)''(u_1 - u_2),\end{aligned}$$

where we used (12) and (13). So after replacing

$$\int_{\tilde{Q}} [a(\nabla u_1, b(u_1)) - a(\nabla u_2, b(u_2))] \cdot \nabla[(\eta_\delta^+)'(u_1 - u_2)\gamma]$$

with

$$\begin{aligned}&\int_{\tilde{Q}} (\eta_\delta^+)'(u_1 - u_2)[a(\nabla u_1, b(u_1)) - a(\nabla u_2, b(u_2))] \cdot \nabla\gamma \\ &\qquad - C \int_{\tilde{Q}} |u_1 - u_2|\, [1 + |\nabla u_1|^r + \Psi(b(u_1)) + \Psi(b(u_2))](\eta_\delta^+)''(u_1 - u_2)\gamma\end{aligned}$$

in (39), all we need to identify the limit $\delta\downarrow 0$ with (36) by dominated convergence is

$$|q_\delta^+(z,z^0)|\leqslant (b(z)-b(z^0))^+, \qquad q_\delta^+(z,z^0)\to (b(z)-b(z^0))^+ \qquad \text{for} \quad \delta\downarrow 0,$$

respectively

$$|q_\delta^-(z,z^0)|\leqslant (b(z^0)-b(z))^+, \qquad q_\delta^-(z,z^0)\to (b(z^0)-b(z))^+ \qquad \text{for} \quad \delta\downarrow 0,$$

which was proved in (29) and (30), and

$$|(\eta_\delta^+)'(z)|\leqslant \operatorname{sign}^+(z), \qquad (\eta_\delta^+)'(z)\to \operatorname{sign}^+(z) \qquad \text{for} \quad \delta\downarrow 0$$

as well as

$$|z|\,(\eta_\delta^+)''(z)\leqslant \sup_{z\in R}|z|\,\eta''(z), \qquad |z|\,(\eta_\delta^+)''(z)\to 0 \qquad \text{for} \quad \delta\downarrow 0.$$

This establishes (36).

We now derive (15) from (36) under the additional assumption that the testfunction γ has zero initial data. Let the nonnegative $\gamma\in C_0^\infty((0,T)\times R^n)$ be given; fix a nonnegative $\phi\in C_0^\infty(R)$ with unit mass. For positive but sufficiently small ε,

$$\gamma_\varepsilon(t_1,t_2,x):=\frac{1}{\varepsilon}\,\phi\left(\frac{t_1-t_2}{\varepsilon}\right)\gamma\left(\frac{t_1+t_2}{2},x\right)$$

is admissible in (36). Thanks to

$$(\partial_{t_1}+\partial_{t_2})\,\gamma_\varepsilon(t_1,t_2,x)=\frac{1}{\varepsilon}\,\phi\left(\frac{t_1-t_2}{\varepsilon}\right)\partial_t\gamma\left(\frac{t_1+t_2}{2},x\right),$$

$$\nabla\gamma_\varepsilon(t_1,t_2,x)=\frac{1}{\varepsilon}\,\phi\left(\frac{t_1-t_2}{\varepsilon}\right)\nabla\gamma\left(\frac{t_1+t_2}{2},x\right),$$

the derivatives of ϕ, which are singular in the limit $\varepsilon\downarrow 0$, cancel. After a change of variables, we obtain from (36)

$$\begin{aligned}\int_R\frac{1}{\varepsilon}\,\phi\left(\frac{\tau}{\varepsilon}\right)\int_Q\{&-(b(u_1)-b(u_2)^\tau)^+\,\partial_t\gamma^{\tau/2}\\ &+\operatorname{sign}^+(u_1-u_2^\tau)[a(\nabla u_1,b(u_1))-a(\nabla u_2,b(u_2))^\tau]\cdot\nabla\gamma^{\tau/2}\\ &+\operatorname{sign}^+(u_1-u_2^\tau)[f(b(u_1))-f(b(u_2)^\tau)]\,\gamma^{\tau/2}]\,d\tau\leqslant 0,\end{aligned}\tag{40}$$

where we use the abbrevation $w^\tau(t) := w(t-\tau)$. Obviously,

$$\lim_{\tau\to 0}\int_Q (b(u_1)-b(u_2)^\tau)^+\,\partial_t\gamma^{\tau/2} = \int_Q (b(u_1)-b(u_2))^+\,\partial_t\gamma. \tag{41}$$

Because of

$$\operatorname{sign}^+(z_1-z_2)[f(b(z_1))-f(b(z_2))] = F(b(z_1), b(z_2)),$$

where

$$F(w_1, w_2) := \operatorname{sign}^+(w_1-w_2)(f(w_1)-f(w_2))$$

is continuous, we obtain

$$\lim_{\tau\to 0}\int_Q \operatorname{sign}^+(u_1-u_2^\tau)[f(b(u_1))-f(b(u_2)^\tau)]\,\gamma^{\tau/2} = \int_Q F(b(u_1), b(u_2))\gamma. \tag{42}$$

The limiting relation

$$\begin{aligned}\lim_{\tau\to 0}\int_Q &\operatorname{sign}^+(u_1-u_2^\tau)[a(\nabla u_1, b(u_1))-a(\nabla u_2, b(u_2))^\tau]\cdot\nabla\gamma^{\tau/2}\\ &=\int_Q \operatorname{sign}^+(u_1-u_2)[a(\nabla u_1, b(u_1))-a(\nabla u_2, b(u_2))]\cdot\nabla\gamma\end{aligned} \tag{43}$$

is seen by writing

$$\begin{aligned}&\operatorname{sign}^+(u_1-u_2^\tau)[a(\nabla u_1, b(u_1))-a(\nabla u_2, b(u_2))^\tau]\\ &\qquad-\operatorname{sign}^+(u_1-u_2)[a(\nabla u_1, b(u_1))-a(\nabla u_2, b(u_2))]\\ &\quad= T_1+T_2+T_3+T_4+T_5,\end{aligned}$$

where

$$\begin{aligned}T_1 &:= [\operatorname{sign}^+(u_1-u_2^\tau)-\operatorname{sign}^+(u_1-u_2)]\\ &\qquad\times[a(\nabla u_1, b(u_2))-a(\nabla u_2, b(u_2))],\\ T_2 &:= \operatorname{sign}^+(u_1-u_2^\tau)[a(\nabla u_1, b(u_1))-a(\nabla u_1, b(u_2^\tau))]\\ &\qquad-\operatorname{sign}^+(u_1-u_2)[a(\nabla u_1, b(u_1))-a(\nabla u_1, b(u_2))],\\ T_3 &:= \operatorname{sign}^+(u_1-u_2^\tau)[a(\nabla u_2, b(u_2))-a(\nabla u_2^\tau, b(u_2))],\\ T_4 &:= -\operatorname{sign}^+(u_1-u_2^\tau)[a(\nabla u_1, b(u_2))-a(\nabla u_1, b(u_2^\tau))],\\ T_5 &:= \operatorname{sign}^+(u_1-u_2^\tau)[a(\nabla u_2^\tau, b(u_2))-a(\nabla u_2^\tau, b(u_2^\tau))].\end{aligned}$$

By Fatou's Lemma,

$$\limsup_{\tau \to 0} \int_Q |T_1|^{r^*} \leqslant \int_{Q \cap \{u_1 = u_2\}} |a(\nabla u_1, b(u_2)) - a(\nabla u_2, b(u_2))|^{r^*} = 0.$$

Because of

$$T_2 = A(\nabla u_1, b(u_1), b(u_2)^\tau) - A(\nabla u_1, b(u_1), b(u_2)),$$

where

$$A(p, w_1, w_2) := \operatorname{sign}^+(w_1 - w_2)[a(p, w_1) - a(p, w_2)]$$

is continuous, we obtain $\lim_{\tau \to 0} \int_Q |T_2|^{r^*} = 0$. Due to

$$|T_3| \leqslant |a(\nabla u_2, b(u_2)) - a(\nabla u_2^\tau, b(u_2))|,$$

and the continuity of $a(p, w)$ in p, we have $\lim_{\tau \to 0} \int_Q |T_3|^{r^*} = 0$. Similarly,

$$|T_4| \leqslant |a(\nabla u_1, b(u_2)) - a(\nabla u_1, b(u_2)^\tau)|$$

and the continuity of $a(p, w)$ in w yield $\lim_{\tau \to 0} \int_Q |T_4|^{r^*} = 0$. T_5 is estimated with help of (13)

$$|T_5| \leqslant C\, |u_2 - u_2^\tau|^{1/r^*} \,[1 + |\nabla u_2^\tau|^r + \Psi(b(u_2))) + \Psi(b(u_2)^\tau)]^{1/r^*},$$

leading to

$$\int_Q |T_5|^{r^2/(r^2-1)} \leqslant C \left\{ \int_Q |u_2 - u_2^\tau|^r \right\}^{1/(r+1)} \times \left\{ \int_Q [1 + |\nabla u_2|^r + \Psi(b(u_2))] \right\}^{1/(r+1)} \to 0$$

for $\delta \downarrow 0$. (40), (41), (42) and (43) now yield

$$\begin{aligned} \int_Q \{ &-(b(u_1) - b(u_2))^+ \,\partial_t \gamma \\ &+ \operatorname{sign}^+(u_1 - u_2)[a(\nabla u_1, b(u_1)) - a(\nabla u_2, b(u_2))] \cdot \nabla \gamma \\ &+ \operatorname{sign}^+(b(u_1) - b(u_2))[f(b(u_1)) - f(b(u_2))]\gamma \} \leqslant 0. \end{aligned} \tag{44}$$

The gap between (44) and (15) obviously is filled by

$$\operatorname*{ess\,lim\,sup}_{t \downarrow 0} \int_\Omega (b(u_1(t)) - b(u_2(t)))^+ \,\gamma(t) \leqslant \int_\Omega (b_1^0 - b_2^0)^+ \,\gamma(0),$$

which immediately follows from Lemma 2 and the inequality

$$(w_1-w_2)^+ \leqslant (w_1-w_1^0)^+ + (w_1^0-w_2^0)^+ + (w_2^0-w_2)^+.$$

This completes the proof of (15).

If (16) holds, (15) in particular yields the one-dimensional weak differential inequality

$$-\int_{(0,T)} \alpha'(t) \int_\Omega (b(u_1(t))-b(u_2(t)))^+ \, dt$$

$$\leqslant \alpha(0) \int_\Omega (b_1^0-b_2^0)^+ + L \int_{(0,T)} \alpha(t) \int_\Omega (b(u_1(t))-b(u_2(t)))^+ \, dt$$

$$\text{for all nonnegative} \quad \alpha \in C_0^\infty((-\infty, T)).$$

An application of Gronwall's Lemma yields (17). This achieves the proof of part (a) of the theorem.

We now consider part (b). The existence of a solution follows from [1, Theorem 1.7]. There, the additional assumption

$$b^0 \in b(R) \qquad \text{a.e. in } \Omega. \tag{45}$$

is made. Let us indicate how to dispense with this assumption. (45) is only used in [1, Lemma 1.5] to show the initial infinitesimal step of the reverse energy inequality for a solution u with initial data b^0, namely

$$\int_\Omega \Psi(b^0) \leqslant \operatorname{ess} \liminf_{t \downarrow 0} \int_\Omega \Psi(b(u(t))). \tag{46}$$

But (46) can be shown without using (45). Indeed, the equation implies that there exists a set $E \subset (0, T)$ of measure zero s.t.

$$b(u(t)) \rightharpoonup b^0 \quad \text{weak* in } L^1(\Omega) \qquad \text{for} \quad t \downarrow 0, \quad t \in R - E.$$

Due to (11), $(b(u(t)))_{t \downarrow 0}$ is uniformly integrable and thus

$$b(u(t)) \rightharpoonup b^0 \quad \text{weakly in } L^1(\Omega) \qquad \text{for} \quad t \downarrow 0, \quad t \in R - E.$$

(46) is then a well-known consequence of the lower semicontinuity and the convexity of Ψ.

Finally, we show uniqueness under condition (16). Let u_1 and u_2 be two solutions with the same initial data b^0. We deduce from (17) that

$$b(u_1) = b(u_2) \qquad \text{a.e. in } Q.$$

The difference of the equation for u_1 and that for u_2 therefore reduces to

$$\int_Q [a(\nabla u_1, b(u_1)) - a(\nabla u_2, b(u_1))] \cdot \nabla\zeta = 0$$

$$\text{for all} \quad \zeta \in L^r((0, T), V) \quad \text{with } \partial_t \zeta \in L^\infty(Q) \qquad \text{and} \qquad \zeta(T) = 0.$$

A standard approximation argument shows that $\zeta = u_1 - u_2$ is also admissible. (12) then yields

$$\nabla u_1 = \nabla u_2 \qquad \text{a.e. in } Q.$$

Because of $\mathscr{H}^{n-1}(S) > 0$, we finally obtain

$$u_1 = u_2 \qquad \text{a.e. in } Q.$$

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